A product estimate, the parabolic Weyl lemma and applications

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Abstract

We prove a product estimate that allows to estimate the quadratic first order nonlinearity of the harmonic map flow in the L^p norm. Then the parabolic analogue of Weyl's lemma for the Lapace operator is established. Both results are applied to prove regularity for the heat flow by parabolic bootstrapping.

1 Introduction and main results

There are two main results. The first result is a product estimate. It allows to estimate the quadratic first order nonlinearity of the harmonic map flow in the L^p norm instead of the $L^{p/2}$ norm – as one expects at first sight. A second application, crucial in [We], is to obtain quadratic estimates sharp enough to prove a refined implicit function theorem.

Throughout we identify $S^1 = \mathbb{R}/\mathbb{Z}$ and think of $v \in C^\infty(\mathbb{R}\times S^1)$ as a smooth function $v : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ which satisfies $v(s,t+1) = v(s,t)$.

Theorem 1.1. *Fix $2 \le p < \infty$. Then there is a positive constant C_p such that*

$$\begin{aligned}&\left(\int_{-T}^0\int_0^1 \left(|\partial_t v|\,|\partial_t w|\right)^p \,dtds\right)^{1/p}\\ &\le C_p\left(\|v\|_p + \|\partial_s v\|_p + \|\partial_t\partial_t v\|_p\right)\left(\|\partial_t w\|_p + \|\partial_t\partial_t w\|_p\right)\end{aligned}$$

for all compactly supported smooth maps $v, w : (-T,0]\times S^1 \to \mathbb{R}^k$.

The second result is a parabolic analogue of the Weyl lemma in the theory of elliptic partial differential equations. This seems to be folklore, known to experts but hidden – if not nonexistent – in the literature. By $\mathbb{H}^-$ we denote the closed lower half plane, the set of all reals (s,t) such that $s \le 0$ and $t \in \mathbb{R}$.

Lemma 1.2 (Parabolic Weyl lemma)**.** *Let $\Omega \subset \mathbb{H}^-$ be an open subset. If $u \in L^1_{loc}(\Omega)$ satisfies*

$$\int_\Omega u\left(-\partial_s\phi - \partial_t\partial_t\phi\right) = 0 \tag{1}$$

for every $\phi \in C_0^\infty(\operatorname{int}\Omega)$, then $u \in C^\infty(\Omega)$ and $\partial_s u - \partial_t\partial_t u = 0$ on Ω.

Notation. For $T > T' > 0$ abbreviate

$$Z = Z_T = (-T, 0] \times S^1, \qquad Z' = Z_{T'} = (-T', 0] \times S^1. \tag{2}$$

To simplify notation we denote the anisotropic Sobolev spaces $W_p^{k,2k}$ by $\mathcal{W}^{k,p}$. More precisely, fix an integer $k \geq 0$, a constant $p \geq 1$, and the domain $\mathbb{R} \times S^1$. Now set $\mathcal{W}^{0,p} = L^p$ and denote by $\mathcal{W}^{1,p}$ the set of all $u \in L^p$ which admit weak derivatives $\partial_s u$, $\partial_t u$, and $\partial_t\partial_t u$ in L^p. For $k \geq 2$ define

$$\mathcal{W}^{k,p} = \{u \in \mathcal{W}^{1,p} \mid \partial_s u, \partial_t u, \partial_t\partial_t u \in \mathcal{W}^{k-1,p}\}$$

where the derivatives are again meant in the weak sense. The associated norm

$$\|u\|_{\mathcal{W}^{k,p}} = \left(\int_{\mathbb{R}\times S^1} \sum_{2\nu+\mu\leq 2k} |\partial_s^\nu \partial_t^\mu u|^p\right)^{1/p}$$

gives $\mathcal{W}^{k,p}$ the structure of a Banach space. Note the difference to (standard) Sobolev space $W^{k,p}$ with norm $\|u\|^p_{\mathcal{W}^{k,p}} = \int_{\mathbb{R}\times S^1} \sum_{\nu+\mu\leq k} |\partial_s^\nu \partial_t^\mu u|^p$.

The parabolic Weyl lemma is the key ingredient to prove part a) of the next theorem. The proof of part b) is based on theorem 4.1 the parabolic analogue of the Calderon-Zygmund inequality.

Theorem 1.3 (Interior regularity)**.** *Fix constants $1 < q < \infty$ and $T > 0$ and an integer $k \geq 0$. Then the following is true.*

a) *If $u \in L^1_{loc}(Z)$ and $f \in \mathcal{W}^{k,q}_{loc}(Z)$ satisfy*

$$\int_Z u\left(-\partial_s\phi - \partial_t\partial_t\phi\right) = \int_Z f\phi \tag{3}$$

for every $\phi \in C_0^\infty((-T, 0) \times S^1)$, then $u \in \mathcal{W}^{k+1,q}_{loc}(Z)$.

b) *For every $0 < T' < T$ there is a constant $c = c(k, q, T - T')$ such that*

$$\|u\|_{\mathcal{W}^{k+1,q}(Z')} \leq c\left(\|\partial_s u - \partial_t\partial_t u\|_{\mathcal{W}^{k,q}(Z)} + \|u\|_{L^q(Z)}\right)$$

for every $u \in C^\infty(\overline{Z})$.

As an application of theorem 1.1 and theorem 1.3, hence lemma 1.2, we prove the following regularity result by parabolic bootstrapping.

Theorem 1.4 (Regularity)**.** *Fix constants $p > 2$, $\mu_0 > 1$, and $T > 0$. Fix a closed smooth submanifold $M \hookrightarrow \mathbb{R}^N$ and a smooth family of vector-valued symmetric bilinear forms $\Gamma : M \to \mathbb{R}^{N\times N\times N}$. Assume that $F : Z \to \mathbb{R}^N$ is a map of class L^p and $u : Z \to \mathbb{R}^N$ is a $\mathcal{W}^{1,p}$ map taking values in M with $\|u\|_{\mathcal{W}^{1,p}} \le \mu_0$ such that the perturbed heat equation*

$$\partial_s u - \partial_t\partial_t u = \Gamma(u)\,(\partial_t u, \partial_t u) + F \tag{4}$$

is satisfied almost everywhere. Then the following is true. For every integer $k \ge 1$ such that $F \in \mathcal{W}^{k,p}(Z)$ and every $T' \in (0,T)$ there is a constant c_k depending on k, p, μ_0, $T - T'$, $\|\Gamma\|_{C^{2k+2}}$, and $\|F\|_{\mathcal{W}^{k,p}(Z)}$ such that

$$\|u\|_{\mathcal{W}^{k+1,p}(Z')} \le c_k.$$

The theorem shows that if F is smooth, then u is smooth on a slightly smaller domain. This result is needed in [We] in the case $F(s,t) = (\operatorname{grad}\mathcal{V}(u(s,\cdot))(t)$ where $\mathcal{V}$ is a smooth function on the free loop space of M.

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2 The product estimate

We prove a version of theorem 1.1 suitable for global analysis.

Proposition 2.1. *Let N be a Riemannian manifold with Levi-Civita connection ∇ and Riemannian curvature tensor R. Fix constants $2 \le p < \infty$ and $c_0 > 0$. Then there is a constant $C = C(p, c_0, \|R\|_\infty) > 0$ such that the following holds. If $u : (a,b] \times S^1 \to N$ is a smooth map such that $\|\partial_s u\|_\infty + \|\partial_t u\|_\infty \le c_0$ then*

$$\left(\int_a^b\int_0^1 \left(|\nabla_t\xi|\,|\nabla_t X|\right)^p\,dtds\right)^{1/p} \le C\,\|\xi\|_{\mathcal{W}^{1,p}}\left(\|\nabla_t X\|_p + \|\nabla_t\nabla_t X\|_p\right)$$

for all smooth compactly supported vector fields ξ and X along u.

Remark 2.2. Proposition 2.1 continues to hold for smooth maps u that are defined on the whole cylinder $\mathbb{R} \times S^1$. In this case the (compact) supports of ξ and X are contained in an interval of the form $(a,b]$.

Lemma 2.3 ([SW, lemma D.4])**.** *Let $x \in C^\infty(S^1, M)$ and $p > 1$. Then*

$$\|\nabla_t\xi\|_p \le \kappa_p\left(\delta^{-1}\|\xi\|_p + \delta\,\|\nabla_t\nabla_t\xi\|_p\right)$$

for $\delta > 0$ and smooth vector fields ξ along x. Here κ_p equals $p/(p-1)$ for $p \le 2$ and it equals p for $p \ge 2$.

Proof of proposition 2.1. The proof has three steps. Step 2 requires $p \geq 2$. Abbreviate $I = (a, b]$ and for $q, r \in [1, \infty]$ consider the norm

$$\|\xi\|_{q;r} := \|\xi\|_{L^q(I, L^r(S^1))}.$$

STEP 1. *Fix reals $\alpha \geq 1$ and $q, r, q', r' \in [\alpha, \infty]$ such that $\frac{1}{q} + \frac{1}{r} = \frac{1}{\alpha}$ and $\frac{1}{q'} + \frac{1}{r'} = \frac{1}{\alpha}$. Then*

$$\|fg\|_\alpha \leq \|f\|_{q';q} \|g\|_{r';r}$$

for all functions $f, g \in C^\infty(I \times S^1)$.

Let $f_s(t) := f(s, t)$. Apply Hölder's inequality twice to obtain

$$\begin{aligned}
\|fg\|^\alpha_{L^\alpha(I \times S^1)} &= \int_a^b \|f_s g_s\|^\alpha_{L^\alpha(S^1)}\, ds \\
&\leq \int_a^b \left(\|f_s\|_{L^q(S^1)} \|g_s\|_{L^r(S^1)} \right)^\alpha ds \\
&= \|uv\|^\alpha_{L^\alpha(I)} \\
&\leq \left(\|u\|_{L^{q'}(I)} \|v\|_{L^{r'}(I)} \right)^\alpha
\end{aligned}$$

where $u(s) := \|f_s\|_{L^q(S^1)}$ and $v(s) := \|g_s\|_{L^r(S^1)}$. This proves Step 1.

STEP 2. *Given p, c_0, and u as in the hypothesis of the lemma. Then there is a constant $c = c(p, c_0) > 0$ such that*

$$\|\nabla_t \xi\|_{\infty;p} \leq c \|\xi\|_{\mathcal{W}^{1,p}}$$

for every smooth compactly supported vector field ξ along $u : I \times S^1 \to N$.

The proof uses the generalized Young inequality: Given reals $a, b, c \geq 0$ and $1 < \alpha, \beta, \gamma < \infty$ such that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1$, then

$$abc \leq \frac{a^\alpha}{\alpha} + \frac{b^\beta}{\beta} + \frac{c^\gamma}{\gamma}. \tag{5}$$

Abbreviate $\xi(s, t)$ by ξ, then integration by parts shows that

$$\begin{aligned}
&\frac{d}{ds} \int_0^1 |\nabla_t \xi(s,t)|^p\, dt \\
&= p \int_0^1 |\nabla_t \xi|^{p-2} \langle \nabla_t \xi, \nabla_t \nabla_s \xi + [\nabla_s, \nabla_t] \xi \rangle\, dt \\
&= -p \int_0^1 \left(\frac{d}{dt} |\nabla_t \xi|^{p-2} \right) \langle \nabla_t \xi, \nabla_s \xi \rangle\, dt - p \int_0^1 |\nabla_t \xi|^{p-2} \langle \nabla_t \nabla_t \xi, \nabla_s \xi \rangle\, dt \\
&\quad + p \int_0^1 |\nabla_t \xi|^{p-2} \langle \nabla_t \xi, R(\partial_s u, \partial_t u) \xi \rangle\, dt \\
&= -p(p-2) \int_0^1 |\nabla_t \xi|^{p-4} \langle \nabla_t \xi, \nabla_t \nabla_t \xi \rangle \langle \nabla_t \xi, \nabla_s \xi \rangle\, dt \\
&\quad - p \int_0^1 |\nabla_t \xi|^{p-2} \left(\langle \nabla_t \nabla_t \xi, \nabla_s \xi \rangle - \langle \nabla_t \xi, R(\partial_s u, \partial_t u) \xi \rangle \right) dt.
\end{aligned}$$

Take the absolute value of the right hand side, apply the generalized Young inequality (5) in the case[1] $p > 2$ with $\alpha = p/(p-2), \beta = p, \gamma = p$, and the standard Young inequality with $\alpha = p/(p-1), \beta = p$ to obtain the inequality

$$\begin{aligned}
&\frac{d}{ds}\int_0^1 |\nabla_t \xi(s,t)|^p \, dt \\
&\le p(p-1)\int_0^1 |\nabla_t \xi|^{p-2} |\nabla_t \nabla_t \xi| \cdot |\nabla_s \xi| \, dt + pc_0^2 \|R\|_\infty \int_0^1 |\nabla_t \xi|^{p-1} |\xi| \, dt \\
&\le p(p-1)\int_0^1 \left(\frac{p-2}{p} |\nabla_t \xi|^p + \frac{1}{p} |\nabla_t \nabla_t \xi|^p + \frac{1}{p} |\nabla_s \xi|^p\right) dt \\
&\quad + pc_0^2 \|R\|_\infty \int_0^1 \left(\frac{p-1}{p} |\nabla_t \xi|^p + \frac{1}{p} |\xi|^p\right) dt \\
&\le C_1 \left(\|\xi_s\|_{L^p(S^1)}^p + \|\nabla_s \xi_s\|_{L^p(S^1)}^p + \|\nabla_t \nabla_t \xi_s\|_{L^p(S^1)}^p\right).
\end{aligned}$$

Here $C_1 > 0$ is a constant depending only on p, c_0, and $\|R\|_\infty$ and $\xi_s(t) := \xi(s,t)$. Note that we used lemma 2.3 to estimate the terms involving $\nabla_t \xi_s$. Now fix $\sigma \in (a,b]$ and integrate this inequality over $s \in (a,\sigma]$ to obtain the estimate

$$\|\nabla_t \xi_\sigma\|_{L^p(S^1)}^p \le c \, \|\xi\|_{\mathcal{W}^{1,p}((a,b]\times S^1)}^p .$$

Here we used compactness of the support of ξ and monotonicity of the integral. Since the right hand side is independent of σ, the proof of Step 2 is complete.

Step 3. *We prove the lemma.*

Define $f(s,t) := |\nabla_t \xi(s,t)|$ and $g(s,t) := |\nabla_t X(s,t)|$. By Step 1 with α, q, and r' equal to p and with $r = q' = \infty$ we have

$$\int_a^b \int_0^1 \left(|\nabla_t \xi(s,t)| \, |\nabla_t X(s,t)|\right)^p \, dtds = \|fg\|_p^p \le \|\nabla_t \xi\|_{\infty;p}^p \, \|\nabla_t X\|_{p;\infty}^p .$$

Now apply Step 2 to the first factor. For the second one we exploit the fact that, since the slices $s \times S^1$ of our domain are compact, there is the Sobolev embedding $W^{1,p}(S^1) \hookrightarrow L^\infty(S^1)$ with constant $\mu = \mu(p) > 0$. It follows that

$$\begin{aligned}
\int_a^b \|\nabla_t X_s\|_{L^\infty(S^1)}^p \, ds &\le \int_a^b \mu^p \, \|\nabla_t X_s\|_{W^{1,p}(S^1)}^p \, ds \\
&= \mu^p \int_a^b \|\nabla_t X_s\|_{L^p(S^1)}^p + \|\nabla_t \nabla_t X_s\|_{L^p(S^1)}^p \, ds.
\end{aligned}$$

This concludes the proof of proposition 2.1. ∎

Proof of theorem 1.1. Proposition 2.1 with $N = \mathbb{R}^k$, $u \equiv const$, $\xi = v$, and $X = w$. ∎

[1] The case $p = 2$ is taken care of by the standard Young inequality.

3 The parabolic Weyl lemma

The structure of proof of lemma 1.2 is the following. First we approximate u via convolution by a family of smooth solutions u_ε which converge to u in L^1. The point is that we convolute over *individual time slices* $s \times \mathbb{R}$ for almost all times s using mollifiers defined on $\mathbb{R}$. (It is also possible to carry over the proof of the original Weyl lemma for the Laplacian using mollifiers supported in $\mathbb{R}^2$. This leads to restrictions and is explained in a separate section below.) On the other hand, given any integer $k \geq 0$, standard local C^k estimates for smooth solutions of the linear homogeneous heat equation in terms of the L^1 norm apply; see [Ev, Sec. 2.3 Thm. 9]. They provide C^k bounds on compact sets in terms of $\|u_\varepsilon\|_1$. Now by Young's convolution inequality $\|u_\varepsilon\|_1 \leq \|u\|_1$. Hence these bounds are uniform in ε. Therefore by Arzela-Ascoli the family u_ε converges in $C^{k-1}_{loc}(\Omega)$ to a map v. Hence $u = v$ by uniqueness of the limit. As this is true for every k and, moreover, every point is contained in a compact subset of Ω it follows that $u \in C^\infty(\Omega)$. Integration by parts then shows that

$$\partial_s u - \partial_t \partial_t u = 0 \tag{6}$$

on the interior of Ω. Since u is C^∞ on Ω this identity continues to hold on Ω.

Proof of lemma 1.2. Every point of Ω is contained in (some translation of) a parabolic set $(-r^2, 0] \times (-r, r)$ whose closure is contained in Ω for some $r > 0$ sufficiently small. Hence we may assume without loss of generality that

$$\Omega = (-r^2, 0] \times (-r, r), \qquad u \in L^1(\Omega).$$

We prove the lemma in nine steps.

1) We introduce appropriate mollifiers: Fix a smooth function $\rho : \mathbb{R} \to [0, 1]$ which is compactly supported in the interval $(-1, 1)$ and satisfies $\int_{\mathbb{R}} \rho = 1$. For $\varepsilon > 0$ consider the mollifier

$$\rho_\varepsilon(t) := \frac{1}{\varepsilon} \rho\left(\frac{t}{\varepsilon}\right).$$

It is compactly supported in the interval $(-\varepsilon, \varepsilon)$ and satisfies $\int_{\mathbb{R}} \rho_\varepsilon = 1$.

2) For almost every $s \in \mathbb{R}$ we define the family $\{\rho_\varepsilon * u_s\}_{\varepsilon>0} \subset C^\infty_0(\mathbb{R})$ and calculate the L^1 norm of its derivatives: Extend u by zero on $\mathbb{R}^2 \setminus \Omega$ and denote the extension again by u. Then $u \in L^1(\mathbb{R}^2)$ and

$$u_s := u(s, \cdot) \in L^1(\mathbb{R})$$

for almost every $s \in \mathbb{R}$. For such s and $\varepsilon > 0$ define

$$(\rho_\varepsilon * u_s)(t) = \int_{\mathbb{R}} \rho_\varepsilon(t - \tau) u_s(\tau)\, d\tau.$$

In this case $\rho_\varepsilon * u_s \in C^\infty_0(\mathbb{R})$,

$$\|\rho_\varepsilon * u_s - u_s\|_{L^1(\mathbb{R})} \to 0 \quad \text{as } \varepsilon \to 0,$$

and $\rho_\varepsilon * u_s$ converges to u_s, as $\varepsilon \to 0$, pointwise almost everywhere on $\mathbb{R}$; see [Jo, App. A]. Moreover, by Young's convolution inequality we obtain that

$$\left\|\rho_\varepsilon * u_s\right\|_{L^1(\mathbb{R})} \leq \left\|\rho_\varepsilon\right\|_{L^1(\mathbb{R})} \left\|u_s\right\|_{L^1(\mathbb{R})} = \left\|u_s\right\|_{L^1(\mathbb{R})}$$

and, more generally, that

$$\begin{aligned}\left\|\frac{d^k}{dt^k}\left(\rho_\varepsilon * u_s\right)\right\|_{L^1(\mathbb{R})} = \left\|\left(\rho_\varepsilon^{(k)} * u_s\right)\right\|_{L^1(\mathbb{R})} &\leq \left\|\rho_\varepsilon^{(k)}\right\|_{L^1(\mathbb{R})} \left\|u_s\right\|_{L^1(\mathbb{R})} \\ &= \frac{\left\|\rho^{(k)}\right\|_{L^1(\mathbb{R})}}{\varepsilon^k} \left\|u_s\right\|_{L^1(\mathbb{R})}\end{aligned}$$

for every positive integer k. Here $\rho^{(k)}$ denotes the k-th derivative of ρ.

3) We prove that for $\varepsilon > 0$ the function defined by

$$u_\varepsilon : \mathbb{R}^2 \to \mathbb{R}, \quad (s,t) \mapsto (\rho_\varepsilon * u_s)(t)$$

is integrable and u_ε converges to u in $L^1(\mathbb{R}^2)$ as $\varepsilon \to 0$. Indeed by step 2)

$$\left\|u_\varepsilon\right\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}} \left\|\rho_\varepsilon * u_s\right\|_{L^1(\mathbb{R})} ds \leq \int_{\mathbb{R}} \left\|u_s\right\|_{L^1(\mathbb{R})} ds = \left\|u\right\|_{L^1(\Omega)}.$$

Now define the family of functions $\{f_\varepsilon : \mathbb{R} \to \mathbb{R}\}_{\varepsilon > 0}$ for almost every s by

$$f_\varepsilon(s) := \left\|\rho_\varepsilon * u_s - u_s\right\|_{L^1(\mathbb{R})}.$$

By the last estimate these functions are integrable

$$\left\|f_\varepsilon\right\|_{L^1(\mathbb{R})} = \left\|u_\varepsilon - u\right\|_{L^1(\mathbb{R}^2)} \leq 2\left\|u\right\|_{L^1(\Omega)}.$$

Moreover, they are dominated almost everywhere by an integrable function g. Namely, by step 2

$$\left|f_\varepsilon(s)\right| \leq 2\left\|u_s\right\|_{L^1(\mathbb{R})} =: g(s), \qquad \left\|g\right\|_{L^1(\mathbb{R})} = 2\left\|u\right\|_{L^2(\Omega)}.$$

Step 2) again shows that $f_\varepsilon \to 0$ as $\varepsilon \to 0$ for almost every s. Hence by the Dominated Convergence Theorem it follows that

$$\begin{aligned}\lim_{\varepsilon\to 0}\left\|u_\varepsilon - u\right\|_{L^1(\mathbb{R}^2)} &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}} \left\|\rho_\varepsilon * u_s - u_s\right\|_{L^1(\mathbb{R})} \, ds \\ &= \int_{\mathbb{R}} \left(\lim_{\varepsilon\to 0} f_\varepsilon\right)(s)\, ds \\ &= 0.\end{aligned}$$

4) The function $u_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ defined in 3) admits integrable weak t-derivatives of all orders: Fix $\varepsilon > 0$ and a positive integer k, then

$$\begin{aligned}\int_{\mathbb{R}^2} u_\varepsilon\, \partial_t^k \psi \, dt\, ds &= \int_{\mathbb{R}^2} \left(\rho_\varepsilon * u_s\right) \partial_t^k \psi \, dt\, ds \\ &= (-1)^k \int_{\mathbb{R}^2} (\rho_\varepsilon^{(k)} * u_s)\, \psi \, dt\, ds\end{aligned}$$

for every $\psi \in C_0^\infty(\mathbb{R}^2)$. Here $\rho_\varepsilon^{(k)}$ denotes the k-th derivative. Moreover, the first step is by definition of u_ε and the second step by integration by parts followed by commuting differentiation and convolution. Next observe that the function $\rho_\varepsilon^{(k)} * u_s$ is integrable. Indeed step 2) shows that

$$\int_{\mathbb{R}} \left\| \rho_\varepsilon^{(k)} * u_s \right\|_{L^1(\mathbb{R})} ds \leq \frac{c_k}{\varepsilon^k} \left\| u \right\|_{L^1(\Omega)}$$

with constant $c_k = c_k(\rho) = \|\partial_t^k \rho\|_{L^1(\mathbb{R})}$. Hence the weak t derivatives of the function $u_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ are integrable and given by

$$\partial_t^k u_\varepsilon(s,t) = (\rho_\varepsilon^{(k)} * u_s)(t).$$

5) Fix $\varepsilon > 0$ and consider the subset

$$\Omega_\varepsilon = (-r^2, 0] \times (-r + \varepsilon, r - \varepsilon) \subset \Omega.$$

We prove by induction that for every integer $k \geq 1$ the weak derivative $\partial_s^k u_\varepsilon$ exists in $L^1(\Omega_\varepsilon)$ and equals $\partial_t^{2k} u_\varepsilon$ almost everywhere on Ω_ε. Here assumption (1) enters.

Case $k = 1$. Straightforward calculation shows that

$$\begin{aligned}
\int_\Omega \psi \, \partial_t \partial_t u_\varepsilon &= \int_{\mathbb{R}^2} \psi(s,t) \left(\int_{\mathbb{R}} \partial_t \partial_t \rho_\varepsilon(t-\tau) u_s(\tau) \, d\tau \right) ds \, dt \\
&= \int_{\mathbb{R}^3} \psi(s,t) \, u(s,\tau) \, \partial_\tau \partial_\tau \rho_\varepsilon(t-\tau) \, d\tau \, ds \, dt \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} u(s,\tau) \, \partial_\tau \partial_\tau \Big(\rho_\varepsilon(t-\tau) \psi(s,t) \Big) \, d\tau \, ds \right) dt \\
&= - \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} u(s,\tau) \, \partial_s \Big(\rho_\varepsilon(t-\tau) \psi(s,t) \Big) \, d\tau \, ds \right) dt \\
&= - \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} \rho_\varepsilon(t-\tau) u_s(\tau) \, d\tau \right) \partial_s \psi(s,t) \, ds \, dt \\
&= - \int_\Omega u_\varepsilon \partial_s \psi
\end{aligned}$$

for every test function $\psi \in C_0^\infty(\operatorname{int} \Omega_\varepsilon)$. This identity means that on $\operatorname{int} \Omega_\varepsilon$, hence on Ω_ε, the weak derivative $\partial_s u_\varepsilon$ exists and equals $\partial_t \partial_t u_\varepsilon$ which is integrable by 4). To prove the identity note that the first and the final step are by definition of u_ε in 3). To obtain the second step we changed the order of integration and applied the chain rule. Steps three and five are obvious. To obtain step four we used assumption (1) and the fact that

$$\phi_t(s,\tau) := \rho_\varepsilon(t-\tau)\psi(s,t)$$

lies in $C_0^\infty(\operatorname{int} \Omega)$ for every $t \in \mathbb{R}$. To prove this assume that $\phi_t(s,\tau) \neq 0$. This means firstly that $\rho_\varepsilon(t-\tau) \neq 0$, hence $\tau \in [-\varepsilon + t, \varepsilon + t]$, and secondly that $\psi(s,t) \neq 0$. Now fix a sufficiently small constant $\delta = \delta(\varepsilon) > 0$ such that

$$\operatorname{supp} \psi \subset [-r^2 + \delta, -\delta] \times [-r + \varepsilon + \delta, r - \varepsilon - \delta] \subset \operatorname{int} \Omega_\varepsilon.$$

It follows that

$$\begin{aligned}(s,\tau) &\in [-r^2+\delta,-\delta]\times[-\varepsilon+(-r+\varepsilon+\delta),\varepsilon+(r-\varepsilon-\delta)]\\ &=[-r^2+\delta,-\delta]\times[-r+\delta,r-\delta]\subset \operatorname{int}\Omega.\end{aligned}$$

Induction step $k\Rightarrow k+1$. The calculation follows the same steps as above. We only indicate the minor differences. Assume that case k is true, then

$$\begin{aligned}\int_\Omega \psi\,\partial_t^{2k+2}u_\varepsilon &= (-1)^{k+1}\int_{\mathbb{R}}\left(\int_{\mathbb{R}^2} u(s,\tau)\,\partial_s^{k+1}\Bigl(\rho_\varepsilon(t-\tau)\psi(s,t)\Bigr)\,d\tau\,ds\right)dt\\ &=(-1)^{k+1}\int_{\mathbb{R}^2} u_\varepsilon(s,t)\,\partial_s^{k+1}\psi(s,t)\,ds\,dt\\ &=-\int_\Omega\left(\partial_s^k u_\varepsilon\right)\partial_s\psi\end{aligned}$$

for every test function $\psi\in C_0^\infty(\operatorname{int}\Omega_\varepsilon)$. Note that to obtain the first step we applied $k+1$ times assumption (1) using that ϕ_t and therefore also its derivatives are in $C_0^\infty(\operatorname{int}\Omega)$. In the final step we used the induction hypothesis to integrate by parts k times the s variable.

6) The function u_ε is smooth on the closure of Ω_ε: Fix $\varepsilon>0$ and positive integers m and ℓ. Then $\partial_t^m\partial_s^\ell u_\varepsilon$ equals $\partial_t^{m+2\ell}u_\varepsilon$ almost everywhere on Ω_ε by 5) and the latter function is integrable by 4). This proves that

$$u_\varepsilon\in\bigcap_{k=1}^\infty W^{k,1}(\Omega_\varepsilon)=C^\infty(\overline{\Omega}_\varepsilon).$$

Moreover, by 5) with $k=1$, each u_ε solves the linear heat equation (6) on $\overline{\Omega}_\varepsilon$.

7) From now on fix a compact subset $Q\subset\Omega$. Then for every positive integer k the family u_ε is uniformly bounded in the Banach space $C^k(Q)$ by a constant $\mu_k=\mu_k(Q)$: To see this consider the compact parabolic set of radius r, height r^2, and top center point $(s,t)\in Q$ given by

$$P_r(s,t):=[s-r^2,s]\times[t-r,t+r].$$

By compactness of Q there is a constant $\varepsilon_0=\varepsilon_0(Q)>0$ such that $Q\subset\Omega_{\varepsilon_0}$ and, moreover, there is a constant $\rho=\rho(\varepsilon_0,Q)>0$ such that

$$P_{2\rho}(s,t)\subset\Omega_{\varepsilon_0}$$

for every point $(s,t)\in Q$. By step 6) each function u_ε with $\varepsilon\in(0,\varepsilon_0)$ is a smooth solution of the linear homogeneous heat equation (6) on the domain Ω_ε and therefore on Ω_{ε_0}. Now given a point $(\sigma,\tau)\in Q$ and a pair of nonnegative integers m,ℓ there is by [Ev, Sec. 2.3 Thm. 9] a constant $c_{m,\ell}(\sigma,\tau)$ such that

$$\max_{P_{\frac{\rho}{2}}(\sigma,\tau)}\left|\partial_t^m\partial_s^\ell v\right|\le\frac{c_{m,\ell}(\sigma,\tau)}{\rho^{m+2\ell+3}}\,\|v\|_{L^1(P_\rho(\sigma,\tau))}$$

for all smooth solutions v of the heat equation (6) in $P_{2\rho}(\sigma,\tau)$. By compactness of Q there are finitely many sets $P_{\rho/2}(\sigma_\nu,\tau_\nu)$ covering Q. Then the corresponding estimates for $v=u_\varepsilon$ and $m,\ell=0,1,\dots,k$ imply that

$$\|u_\varepsilon\|_{C^k(Q)} \le \alpha \|u_\varepsilon\|_{L^1(\mathbb{R}^2)} \le \alpha \|u\|_{L^1(\Omega)}$$

for every $\varepsilon\in(0,\varepsilon_0)$ and where the constant $\alpha>0$ depends only on the compact set Q (since ρ eventually depends on Q only). Here the second inequality is proved in step 3).

8) We prove that $u\in C^\infty(Q)$. In the setting of step 7) the Arzela-Ascoli theorem for each k together with choosing a diagonal subsequence yields existence of a sequence $\varepsilon_k\to 0$, as $k\to\infty$, and a smooth function $\hat{u}$ defined on Q such that $u_{\varepsilon_k}\to\hat{u}$ in $C^\infty(Q)$, as $k\to\infty$. On the other hand, the sequence u_{ε_k} converges to u in $L^1(Q)$ by step 3). Hence $u=\hat{u}$ by uniqueness of limits.

9) We prove lemma 1.2. Since every point of Ω is contained in a compact subset Q and $u\in C^\infty(Q)$ by step 8), the function u is smooth on Ω. To prove the identity $\partial_s u-\partial_t\partial_t u=0$ on Ω assume by contradiction that this identity is violated at a point (s_*,t_*) of Ω. There are two cases.
If (s_*,t_*) is in the interior of Ω, then by smoothness of u there is a sufficiently small open neighborhood U of (s_*,t_*) in Ω and a function $\phi\in C_0^\infty(U,[0,1])$ with $\phi(s_*,t_*)=1$ such that assumption (1) fails. (For instance, if $c>0$ is the value of the function $\partial_s u-\partial_t\partial_t u$ at the point (s_*,t_*), let U be the subset of Ω on which $\partial_s u-\partial_t\partial_t u>c/2$.)
If (s_*,t_*) is in the boundary $0\times(-r,r)$ of Ω, the former argument works for an interior point of Ω sufficiently close to (s_*,t_*). Existence of such an interior point uses again smoothness of u on Ω. This proves the parabolic Weyl lemma. $\square$

The heat ball approach

A natural first try to prove lemma 1.2 is to carry over the proof of the original Weyl lemma for the Laplacian; see e.g. [GT, Jo]). This works, but with two restrictions. Firstly, the set Ω should be open in $\mathbb{R}^2$ and, secondly, the function u should be locally L^q integrable over Ω for some $q>3$.

The original proof is based on the fact that harmonic functions are characterized by their mean value property with respect to balls or spheres. There is a similar statement for solutions to the heat equation. However, in the corresponding parabolic mean value equalities a weight factor different from one appears and this eventually leads to the restriction $q>3$. A further difference is that balls and spheres over which the means are taken are replaced by heat balls and their boundaries, respectively. The parabolic mean value property with respect to boundaries is due to Fulks [Fu] and with respect to heat balls it is due to Watson [Wa]. Here it is required that Ω is open in $\mathbb{R}^2$.

Recall that the *fundamental solution to the heat equation* is given by

$$\Phi(s,t) := \begin{cases} \dfrac{1}{\sqrt{4\pi s}}\, e^{-\dfrac{t^2}{4s}} & , s>0,\ t\in\mathbb{R}, \\ 0 & , s<0,\ t\in\mathbb{R}. \end{cases} \tag{7}$$

For $r > 0$ we denote by $E_r = E_r(0,0)$ the area which is enclosed by the level set determined by the identity

$$\Phi(-s,-t) = \frac{1}{2r\sqrt{\pi}}.$$

This level set is parametrized by

$$t(s) = \pm\sqrt{2s \ln \frac{-s}{r^2}}, \quad s \in (-r^2, 0).$$

Think of it as resembling an ellipse in the plane such that the origin is located at the 'north pole'. For general base point $(s,t) \in \mathbb{R}^2$ the set $E_r(s,t)$ is defined by translation. These sets are called *heat balls of "radius" r*. Following Watson [Wa] we call a function u defined on an open subset $\Omega \subset \mathbb{R}^2$ a *temperature* if $\partial_t \partial_t u$ and $\partial_s u$ are continuous functions on Ω and the heat equation $\partial_s u - \partial_t \partial_t u = 0$ is satisfied pointwise on Ω. (Note that temperatures are automatically C^∞ smooth; see e.g. [Ev, Sec. 2.3 Thm. 8].)

Theorem 3.1 ([Wa] § 10 Cor. 1)**.** *Let u be a continuous function on an open subset $\Omega \subset \mathbb{R}^2$. Then the following are equivalent.*

(a) *The function u is a temperature.*

(b) *At every point $(s,t) \in \Omega$ the weighted mean value equality for u holds*

$$u(s,t) = \frac{1}{8\sqrt{\pi} \cdot r} \int_{E_r(s,t)} \frac{(t-\tau)^2}{(s-\sigma)^2}\, u(\sigma,\tau)\, d\tau\, d\sigma$$

whenever $\overline{E_r(s,t)} \subset \Omega$.

We sketch the proof of the parabolic Weyl lemma (subject to the two restrictions mentioned above) along the lines of the original proof for the Laplacian. Since smoothness is a local property we may assume without loss of generality that $\Omega \subset \mathbb{R}^2$ is bounded. Moreover, we extend u by zero to $\mathbb{R}^2 \setminus \Omega$ without change of notation. Hence $u \in L^q(\Omega)$ for some $q > 3$. The main idea is to mollify the given weak solution u to obtain a family $\{u_r\} \subset C_0^\infty(\mathbb{R}^2)$ of smooth functions converging in L^1, hence almost everywhere, to u. Here we use a family of mollifiers $\{\rho_r\}$ which are compactly supported in the heat ball $E_r \subset \mathbb{R}^2$ and set $u_r = \rho_r * u$ where $*$ denotes convolution. Assumption (1) is then used to show that each function u_r is a temperature on a slightly smaller set $\Omega_r \subset \Omega$ which by definition consists of all points $(s,t) \in \Omega$ such that the closure of the heat ball $E_r(s,t)$ is contained in Ω. Hence each $u_r : \Omega_r \to \mathbb{R}$ satisfies the weighted mean value equality of theorem 3.1. On the other hand, the family $\{u_r\}$ is uniformly bounded – here the restriction $q > 3$ arises – and equicontinuous. Hence by Arzela-Ascoli it converges in C^0 to a continuous function v as $r \to 0$. Since the functions u_r satisfy the mean value equality, so does their C^0 limit v, and therefore v is a temperature by Watson's result theorem 3.1. But $v = u$, since $\{u_r\}$ converges to u almost everywhere.

As it is essentially the only point where the proof of the original Weyl lemma for the Laplacian differs we provide the details of uniform boundedness of the family $\{u_r\}$ on Ω_R. More precisely, fix a constant $R > 0$ and restrict r to the interval $(0, R/2)$. Then

$$\Omega_R \subset \Omega_{R/2} \subset \Omega_r \subset \Omega, \qquad \overline{E_{R/2}(s,t)} \subset \Omega_{R/2} \quad \forall (s,t) \in \Omega_R.$$

Hence by theorem 3.1 each temperature u_r satisfies the mean value equality on all heat balls with base point in Ω_R and radius less or equal to $R/2$. To see that the family $\{u_r\}_{r \in (0,R/2)}$ is uniformly bounded on Ω_R fix a point $(s_0, t_0) \in \Omega_R$. Then by the mean value equality for the temperature u_r over the heat ball $E_{R/2}(s_0, t_0)$ it follows that

$$\begin{aligned}
|u_r(s_0,t_0)| &\le \frac{1}{4\sqrt{\pi}R} \int_{E_{R/2}(s_0,t_0)} \frac{(t_0-\tau)^2}{(s_0-\sigma)^2} \left|u_r(\sigma,\tau)\right| \; d\tau d\sigma \\
&= \frac{1}{4\sqrt{\pi}R} \int_{E_{R/2}(0,0)} \frac{t^2}{s^2} \left|u_r(s+s_0, t+t_0)\right| \; dtds \\
&\le \frac{1}{4\sqrt{\pi}R} \left\| t^2 s^{-2} \right\|_{L^p(E_{R/2})} \left\| u_r \right\|_{L^q(\mathbb{R}^2)} \\
&\le c_{q,R} \left\| u \right\|_{L^q(\Omega)} .
\end{aligned}$$

To obtain step two we introduced new variables $t = \tau - t_0$ and $s = \sigma - s_0$. In step three we use Hölder's inequality with $1/p + 1/q = 1$ and $p, q > 1$. Since the weight function $t^2 s^{-2}$ is not bounded on $E_{R/2}$ we can't get away with pulling out the sup norm as in the proof of the original Weyl lemma for the Laplacian where the weight is one. In the last step we used that

$$\left\| u_r \right\|_{L^q(\mathbb{R}^2)} = \left\| \rho_r * u \right\|_{L^q(\mathbb{R}^2)} \le \left\| \rho_r \right\|_{L^1(\mathbb{R}^2)} \left\| u \right\|_{L^q(\mathbb{R}^2)} = \left\| u \right\|_{L^q(\Omega)}$$

by Young's convolution inequality. Moreover, the constant $c_{q,R}$ is given by $\|t^2 s^{-2}\|_{L^p(E_{R/2})} / 4\sqrt{\pi}R$ with $p = \frac{q}{q-1}$. To see that it is finite observe that

$$\begin{aligned}
\left\| t^2 s^{-2} \right\|^p_{L^p(E_1)} &= \frac{2^{p+\frac{3}{2}}}{2p+1} \int_{-1}^{0} \frac{(s \ln(-s))^{p+\frac{1}{2}}}{(-s)^{2p}} ds \\
&= \frac{2^{p+\frac{3}{2}}}{2p+1} \int_0^\infty x^{p+\frac{1}{2}} e^{-x(\frac{3}{2}-p)} dx \\
&= \frac{2^{p+\frac{3}{2}}}{2p+1} \frac{\Gamma(p+\frac{3}{2})}{\left(\frac{3}{2}-p\right)^{p+\frac{3}{2}}}.
\end{aligned}$$

Here we used the change of variables $x = -\log(-s)$ in the second step, the last step is valid whenever $-\frac{3}{2} < p < \frac{3}{2}$, and Γ denotes the gamma function. The earlier use of Hölder's inequality further restricts p to the interval $(1, \frac{3}{2})$ and this is equivalent to $q = \frac{p}{p-1} > 3$. It remains to replace the unit heat ball E_1 by $E_{R/2}$. This leads to a further constant which depends only on R and p.

4 Local regularity

The parabolic analogue of the Calderon-Zygmund inequality is the following fundamental L^p estimate. It is used in the proof of theorem 4.2 on local regularity and it implies the interior estimates of theorem 4.4 by induction.

Theorem 4.1 (Fundamental L^p estimate). *For every $p > 1$, there is a constant $c = c(p) > 0$ such that*

$$\|\partial_s v\|_p + \|\partial_t \partial_t v\|_p \leq c \|\partial_s v - \partial_t \partial_t v\|_p$$

for every $v \in C_0^\infty(\mathbb{R}^2)$. The same statement is even true for the domain $\mathbb{H}^-$.

Proof. A proof for $\mathbb{R}^2$ is given in [SW, theorem C.2] by the Marcinkiewicz-Mihlin multiplier method. In the case of the lower half plane $\mathbb{H}^-$ choose a compactly supported smooth function v on $\mathbb{H}^-$ and constants $T > 0$ and $a < b$ such that $\operatorname{supp} u \subset (-T/2, 0] \times (a, b)$. Then [Li, proposition 7.11] with $n = 1$, $A^{11} = 1$, $\lambda = \Lambda = 1$, the cube $K_0 = (-T/2, 0] \times (a, b)$ in $(-T, 0) \times \mathbb{R}$, and the function $f = \partial_s u - \partial_t \partial_t u$ proves the statement. Note that the case $\mathbb{H}^-$ implies the case $\mathbb{R}^2$ by translation. $\square$

Theorem 4.2 (Local regularity). *Fix a constant $1 < q < \infty$, an integer $k \geq 0$, and an open subset $\Omega \subset \mathbb{H}^-$. Then the following is true.*

a) *If $u \in L^1_{loc}(\Omega)$ and $f \in \mathcal{W}^{k,q}_{loc}(\Omega)$ satisfy*

$$\int_\Omega u \left(-\partial_s \phi - \partial_t \partial_t \phi\right) = \int_\Omega f \phi \tag{8}$$

for every $\phi \in C_0^\infty(\operatorname{int} \Omega)$, then $u \in \mathcal{W}^{k+1,q}_{loc}(\Omega)$.

b) *If $u \in L^1_{loc}(\Omega)$ and $f, h \in \mathcal{W}^{k,q}_{loc}(\Omega)$ satisfy*

$$\int_\Omega u \left(-\partial_s \phi - \partial_t \partial_t \phi\right) = \int_\Omega f \phi - \int_\Omega h \, \partial_t \phi \tag{9}$$

for every $\phi \in C_0^\infty(\operatorname{int} \Omega)$, then u and $\partial_t u$ are in $\mathcal{W}^{k,q}_{loc}(\Omega)$.

Here $\operatorname{int} \Omega$ denotes the interior of Ω. While part b) is not needed in this text it is used in [We] to prove regularity of the solutions of the *linearized* heat equation. For convenience of the reader we recall Poincaré's inequality and its proof. It is used to prove theorem 4.2 and theorem 4.4.

Lemma 4.3 (Poincaré's inequality). *Fix constants $q \geq 1$ and $r > 0$. Then*

$$\|\varphi\|_q \leq 2r \, \|\partial_t \varphi\|_q$$

for every $\varphi \in C_0^\infty((-r, r))$.

Proof. For such φ it holds that $\varphi(-r) = 0$ and hence $\varphi(t) = \int_{-r}^{t} \partial_t \varphi(\tau)\, d\tau$ by the fundamental theorem of calculus. This implies that

$$|\varphi(t)| \leq \int_{-r}^{t} |\partial_t \varphi(\tau)|\ d\tau \leq \int_{-r}^{r} 1 \cdot |\partial_t \varphi(\tau)|\ d\tau \leq (2r)^{1/p} \left\|\partial_t \varphi\right\|_q$$

where the last step uses Hölder's inequality with $1/q + 1/p = 1$. Therefore

$$|\varphi(t)|^q \leq (2r)^{q-1} \left\|\partial_t \varphi\right\|_q^q$$

and integration over $t \in (-r, r)$ concludes the proof of the lemma. $\square$

Proof of theorem 4.2 . Since any given compact subset Q of Ω can be covered by finitely many parabolic rectangles whose closure is contained in Ω, we may assume without loss of generality that $\Omega = (-r^2, 0] \times (-r, r)$ for $r > 0$.
ad a) The proof consists of four steps.

I) Fix two open subsets Ω' and U of $\Omega = (-r^2, 0] \times (-r, r)$ such that the closure of Ω' is contained in U and the closure of U is contained in Ω. Fix a smooth compactly supported cutoff function $\beta : \Omega \to [0, 1]$ such that $\beta = 1$ on U. Then βf is compactly supported and $\mathcal{W}^{k,q}$ integrable over Ω. Now approximate βf in $\mathcal{W}^{k,q}(\Omega)$ through a sequence $(f_i) \subset C_0^\infty(\Omega)$, i.e.

$$\|f_i - \beta f\|_{\mathcal{W}^{k,q}(\Omega)} \longrightarrow 0, \qquad \text{as } i \to \infty.$$

II) Each smooth problem

$$(\partial_s - \partial_t \partial_t) u_i = f_i \tag{10}$$

with $f_i \in C_0^\infty(\Omega)$ admits a unique solution $u_i \in C_0^\infty(\Omega)$; see e.g. [Li, Thm. 5.6]. We prove below that the sequence of solutions u_i is a Cauchy sequence in $\mathcal{W}^{k+1,q}(\Omega)$. Therefore it admits a unique limit $\hat{u} \in \mathcal{W}^{k+1,q}(\Omega)$. Now the limit $\hat{u}$ solves the identity $(\partial_s - \partial_t \partial_t)\hat{u} = \beta f$ almost everywhere on Ω as can be seen as follows: The sequence $\partial_s u_i - \partial_t \partial_t u_i$ converges to $\partial_s \hat{u} - \partial_t \partial_t \hat{u}$ in L^q, since u_i is a Cauchy sequence in $\mathcal{W}^{k+1,q}(\Omega)$, and the sequence f_i converges to βf by step I). Uniqueness of the limit then proves equality in $L^q(\Omega)$.
It remains to prove that the sequence u_i is Cauchy. All norms are with respect to the domain Ω. Note that

$$\left\|u_i - u_j\right\|_q \leq 2r \left\|\partial_t (u_i - u_j)\right\|_q \leq (2r)^2 \left\|\partial_t \partial_t (u_i - u_j)\right\|_q$$

Here the first inequality follows by integrating Poincaré's inequality (lemma 4.3) for $\varphi(t) = u_i(s,t) - u_j(s,t)$ over $s \in (-r^2, 0)$. The second inequality follows similarly. Now use equation (10) to obtain that

$$\left\|u_i - u_j\right\|_q \leq (2r)^2 \left(\left\|\partial_s (u_i - u_j)\right\|_q + \left\|f_i - f_j\right\|_q \right).$$

More generally, there is a constant $C = C(k, r)$ such that

$$\|u_i - u_j\|_{\mathcal{W}^{k+1,q}} \leq C \left(\left\|\partial_s^{k+1} (u_i - u_j)\right\|_q + \|f_i - f_j\|_{\mathcal{W}^{k,q}} \right).$$

for all i and j. This follows by inspecting the left hand side term by term replacing any two t-derivatives by one s-derivative and the error term f_i according to equation (10). If an odd number of t-derivatives appears then use lemma 4.3 to obtain an even number. Now the fundamental L^p estimate theorem 4.1 with constant $c = c(q)$ and function $v = \partial_s^k(u_i - u_j)$ asserts that

$$\begin{aligned}\|\partial_s^{k+1}(u_i - u_j)\|_q &\le c\|(\partial_s - \partial_t\partial_t)\partial_s^k(u_i - u_j)\|_q \\ &= c\|\partial_s^k(f_i - f_j)\|_q \\ &\le c\|f_i - f_j\|_{\mathcal{W}^{k,q}}.\end{aligned}$$

Here we used again equation (10). Next use the approximation of βf in step I) to obtain that the sequence u_i in $\mathcal{W}^{k,q}(\Omega)$ is Cauchy, namely

$$\|f_i - f_j\|_{\mathcal{W}^{k,q}} \le \|f_i - \beta f\|_{\mathcal{W}^{k,q}} + \|\beta f - f_j\|_{\mathcal{W}^{k,q}} \longrightarrow 0, \qquad \text{as } i, j \to \infty.$$

III) The restriction of $\hat{u} - u$ to the open subset $U \subset \Omega$ is a weak solution of the homogeneous problem. More precisely, it is true that

$$\begin{aligned}\int_U (\hat{u} - u)(-\partial_s\phi - \partial_t\partial_t\phi) &= \int_U (\partial_s\hat{u} - \partial_t\partial_t\hat{u})\phi - \int_U u(-\partial_s\phi - \partial_t\partial_t\phi) \\ &= \int_U (\partial_s\hat{u} - \partial_t\partial_t\hat{u} - \beta f)\phi \\ &= 0\end{aligned}$$

for every test function $\phi \in C_0^\infty(\operatorname{int} U)$. Here the first step is by integration by parts using step II) and the second step is by assumption (8) and the fact that $f = \beta f$ on U. The last step uses the identity in step II).

IV) The difference $\hat{u} - u$ is in $L^1(U)$ by step II) and assumption on u. Hence by the parabolic Weyl lemma 1.2 the function $F := \hat{u} - u$ is smooth on U. Together with the fact that $\hat{u} \in \mathcal{W}^{k+1,q}(\Omega)$ proved in step II) this shows that $u = \hat{u} - F$ is of class $\mathcal{W}^{k+1,q}$ on each bounded open subset of U, in particular on Ω'. This proves part a) of theorem 4.2.

ad b) The proof takes four further steps.

V) Let the sets Ω' and U, the cutoff function β, and the sequence $(f_i) \subset C_0^\infty(\Omega)$ be as in step I). Approximate the compactly supported function βh in $\mathcal{W}^{k,q}(\Omega)$ through a sequence $(h_i) \subset C_0^\infty(\Omega)$. Now as in steps II) and III) each smooth problem

$$(\partial_s - \partial_t\partial_t)v_i = h_i \tag{11}$$

admits a unique solution $v_i \in C_0^\infty(\Omega)$ and the sequence (v_i) is Cauchy in $\mathcal{W}^{k+1,q}(\Omega)$ with unique limit $\hat{v}$ which solves the identity $(\partial_s - \partial_t\partial_t)\hat{v} = \beta h$ almost everywhere on Ω.

VI) Observe that the sequences

$$w_i := u_i + \partial_t v_i, \qquad \partial_t w_i = \partial_t u_i + \partial_t\partial_t v_i,$$

converge in $\mathcal{W}^{k,q}(\Omega)$ to the limits

$$\hat{w} = \hat{u} + \partial_t\hat{v}, \qquad \partial_t\hat{w} = \partial_t\hat{u} + \partial_t\partial_t\hat{v},$$

respectively. Moreover, each w_i satisfies the identity $(\partial_s - \partial_t\partial_t)w_i = f_i + \partial_t h_i$ on Ω. Integration by parts then shows that

$$\int_\Omega w_i\,(-\partial_s - \partial_t\partial_t)\,\phi = \int_\Omega f_i\phi - \int_\Omega h_i\partial_t\phi$$

for every $\phi \in C_0^\infty(\operatorname{int}\Omega)$. Taking the limit $i \to \infty$ we obtain that

$$\int_\Omega \hat{w}\,(-\partial_s - \partial_t\partial_t)\,\phi = \int_\Omega \beta f\phi - \int_\Omega \beta h\,\partial_t\phi \tag{12}$$

for every $\phi \in C_0^\infty(\operatorname{int}\Omega)$.

VII) The restriction of $\hat{w} - u$ to the open subset U of Ω is a weak solution of the homogeneous problem, meaning that

$$\begin{aligned}\int_U (\hat{w} - u)(-\partial_s\phi - \partial_t\partial_t\phi) &= \int_U \hat{w}(-\partial_s - \partial_t\partial_t)\phi - \int_U u(-\partial_s\phi - \partial_t\partial_t\phi)\\ &= \int_U (\beta f\phi - \beta h\,\partial_t\phi) - \int_U (f\phi - h\,\partial_t\phi)\\ &= 0\end{aligned}$$

for every test function $\phi \in C_0^\infty(\operatorname{int}U)$. Here step two uses the identity (12) for $\hat{w}$ and assumption (9) on u. Step three is true since $\beta = 1$ on U.

VIII) Note that the difference $\hat{w} - u$ is in $L^1(U)$ by step VI) and assumption on u. Hence by the parabolic Weyl lemma 1.2 the function $G := \hat{w} - u$ is smooth on U. Since $\hat{w} \in \mathcal{W}^{k,q}(\Omega)$ by step VI), this shows that $u = \hat{w} - G$ is of class $\mathcal{W}^{k,q}$ on each bounded open subset of U. Since also $\partial_t\hat{w} \in \mathcal{W}^{k,q}(\Omega)$ by step VI), the function $\partial_t u = \partial_t\hat{w} - \partial_t G$ is of class $\mathcal{W}^{k,q}$ on each bounded open subset of U, in particular on Ω'. This concludes the proof of theorem 4.2. □

Interior estimates

Theorem 4.4 (Interior estimates)**.** *Fix an integer $k \ge 0$ and constants $1 < q < \infty$ and $0 < r < R$. Define $\Omega_r = (-r^2, 0] \times (-r, r)$. Then there is a constant $c = c(k, q, R - r)$ such that*

$$\|u\|_{\mathcal{W}^{k+1,q}(\Omega_r)} \le c\left(\|\partial_s u - \partial_t\partial_t u\|_{\mathcal{W}^{k,q}(\Omega_R)} + \|u\|_{L^q(\Omega_R)} + \|\partial_t u\|_{L^q(\Omega_R)}\right) \tag{13}$$

for every $u \in C^\infty(\overline{\Omega_R})$.

Proof. The proof is by induction on k.
Case $k = 0$. Fix a smooth compactly supported cutoff function $\beta : \Omega_R \to [0, 1]$ such that $\beta = 1$ on Ω_r. Then

$$\begin{aligned}&\|u\|_{\mathcal{W}^{1,q}(\Omega_r)}\\ &\le \|\beta u\|_{L^q(\Omega_R)} + \|\partial_t(\beta u)\|_{L^q(\Omega_R)} + \|\partial_t\partial_t(\beta u)\|_{L^q(\Omega_R)} + \|\partial_s(\beta u)\|_{L^q(\Omega_R)}\\ &\le 2R(1 + 2R)\,\|\partial_t\partial_t(\beta u)\|_{L^q(\Omega_R)} + \|\partial_s(\beta u)\|_{L^q(\Omega_R)}\\ &\le c\,\|(\partial_s - \partial_t\partial_t)\beta u\|_{L^q(\Omega_R)}\\ &\le c\,\|(\partial_s - \partial_t\partial_t)u\|_{L^q(\Omega_R)} + C\left(\|u\|_{L^q(\Omega_R)} + \|\partial_t u\|_{L^q(\Omega_R)}\right)\end{aligned}$$

where $c = c_q\,(1 + 2R(1 + 2R))$ with c_q being the constant in theorem 4.1 and

$$C = \|\partial_s \beta\|_\infty + \|\partial_t \partial_t \beta\|_\infty + 2\,\|\partial_t \beta\|_\infty .$$

The first step uses the fact that $\beta = 1$ on Ω_r, the definition of the $\mathcal{W}^{1,q}$ norm, and monotonicity of the integral. To obtain step two we fixed s and applied Poincaré's inequality lemma 4.3 to the functions $\beta u, \partial_t(\beta u) \in C_0^\infty(-R, R)$, then we integrated over $s \in (-R^2, 0]$. Step three is by theorem 4.1.

Induction step $k-1 \Rightarrow k$. Fix $k \geq 1$. It suffices to estimate the $\mathcal{W}^{k+1,q}$ norms of u, $\partial_t u$, $\partial_t \partial_t u$, and $\partial_s u$ individually by the right hand side of (13). We provide details for the least trivial term and leave the others as an exercise. Fix constants $r < r_1 < r_2 < R$. Then by the induction hypothesis in the case $k-1$ for the pair of sets $\Omega_r \subset \Omega_{r_1}$ and the function $v = \partial_s u$ we obtain that

$$\begin{aligned}
&\|\partial_s u\|_{\mathcal{W}^{k,q}(\Omega_r)} \\
&\leq c_1 \left(\|(\partial_s - \partial_t \partial_t)\partial_s u\|_{\mathcal{W}^{k-1,q}(\Omega_{r_1})} + \|\partial_s u\|_{L^q(\Omega_{r_1})} + \|\partial_t \partial_s u\|_{L^q(\Omega_{r_1})} \right) \\
&\leq c_1 \left(\|(\partial_s u - \partial_t \partial_t) u\|_{\mathcal{W}^{k,q}(\Omega_R)} + \|u\|_{\mathcal{W}^{1,q}(\Omega_{r_1})} + \|\partial_t u\|_{\mathcal{W}^{1,q}(\Omega_{r_1})} \right)
\end{aligned}$$

for some constant $c_1 = c_1(k-1, q, r_1 - r)$. To deal with the last term in the sum we apply the case $k = 0$ for the pair of sets $\Omega_{r_1} \subset \Omega_{r_2}$ and the function $v = \partial_t u$ to obtain that

$$\begin{aligned}
&\|\partial_t u\|_{\mathcal{W}^{1,q}(\Omega_{r_1})} \\
&\leq c_2 \left(\|(\partial_s - \partial_t \partial_t)\partial_t u\|_{L^q(\Omega_{r_2})} + \|\partial_t u\|_{L^q(\Omega_{r_2})} + \|\partial_t \partial_t u\|_{L^q(\Omega_{r_2})} \right) \\
&\leq c_2 \left(\|(\partial_s u - \partial_t \partial_t) u\|_{\mathcal{W}^{k,q}(\Omega_R)} + \|\partial_t u\|_{L^q(\Omega_R)} + \|u\|_{\mathcal{W}^{1,q}(\Omega_{r_2})} \right)
\end{aligned}$$

for some constant $c_2 = c_2(q, r_2 - r_1)$. It remains to estimate the last term in the sum. We apply again the case $k = 0$, but now for the pair of sets $\Omega_{r_2} \subset \Omega_R$ and the function u to obtain that

$$\|u\|_{\mathcal{W}^{1,q}(\Omega_{r_2})} \leq c_3 \left(\|(\partial_s - \partial_t \partial_t) u\|_{L^q(\Omega_R)} + \|u\|_{L^q(\Omega_R)} + \|\partial_t u\|_{L^q(\Omega_R)} \right)$$

for some constant $c_3 = c_3(q, R - r_2)$. This proves theorem 4.4. □

Proof of theorem 1.3. a) Suppose the parabolic rectangle $\Omega = (\sigma - r^2, \sigma] \times (\tau - r, \tau + r)$ is contained in the cylinder $Z_T = (-T, 0] \times S^1$. Then the assumptions of theorem 4.2 a) are satisfied for the restrictions of u and f to Ω and therefore $u \in \mathcal{W}^{k+1,q}_{loc}(\Omega)$. Now every compact subset of Z_T can be covered by finitely many parabolic rectangles. Hence u is locally $\mathcal{W}^{k+1,q}$ integrable on Z_T.

b) Induction over k based on theorem 4.4 and a covering argument by parabolic rectangles proves b). □

5 Parabolic bootstrapping

In this section we establish uniform Sobolev bounds for strong solutions u of the heat equation (14) by parabolic bootstrapping. This immediately implies theorem 1.4. In order to deal with the heat equation's quadratic nonlinearity in $\partial_t u$ we first prove in lemma 5.1 apriori continuity of $\partial_t u$. Then the heat equation can be treated like a linear equation in the crucial first step $\ell = 1$ of the parabolic bootstrap.

In this section we fix a closed smooth submanifold $M \hookrightarrow \mathbb{R}^N$ and a smooth family of vector-valued symmetric bilinear forms $\Gamma : M \to \mathbb{R}^{N\times N\times N}$. Recall that the cylinders $Z = Z_T$ and $Z' = Z_{T'}$ are defined by (2).

Lemma 5.1 (Apriori continuity of $\partial_t u$)**.** *Fix constants $p > 2$, $\mu_0 > 1$, and $T > 0$. Fix a map $F : Z \to \mathbb{R}^N$ such that F and $\partial_t F$ are of class L^p. Assume that $u : Z \to \mathbb{R}^N$ is a $\mathcal{W}^{1,p}$ map taking values in M with $\|u\|_{\mathcal{W}^{1,p}} \le \mu_0$ and such that the perturbed heat equation*

$$\partial_s u - \partial_t\partial_t u = \Gamma(u)\,(\partial_t u, \partial_t u) + F \tag{14}$$

is satisfied almost everywhere. Then $\partial_t u$ is continuous. More precisely, for every $T' \in (0, T)$ there is a constant $c = c(p, \mu_0, T, T', \|\Gamma\|_{C^1})$ such that

$$\|\partial_t u\|_{C^0(Z')} \le c\left(1 + \|\partial_t F\|_{L^p(Z)}\right).$$

Note that by the Sobolev embedding theorem the assumption $p > 2$ guarantees that the $\mathcal{W}^{1,p}$ map u is continuous. Hence it makes sense to specify that u takes values in the submanifold M of $\mathbb{R}^N$. Abbreviate $\mathcal{W}^{k,p}(Z) = \mathcal{W}^{k,p}(Z, \mathbb{R}^N)$.

Remark 5.2. Since the proof of lemma 5.1 relies heavily on the product estimate theorem 1.1 it seems unlikely that the assumption $u \in \mathcal{W}^{1,p}$ can be weakened to $u \in W^{1,p}$ – unless we also replace the assumption $p > 2$ by $p > 3$.

Proposition 5.3. *Under the assumptions of lemma 5.1 the following is true for every integer $k \ge 1$ such that F and $\partial_t F$ are in $\mathcal{W}^{k-1,p}(Z)$ and every constant $T' \in (0, T)$.*

(i) *There is a constant a_k depending on p, μ_0, T, T', $\|\Gamma\|_{C^{2k+2}}$, and the $\mathcal{W}^{k-1,p}(Z)$ norms of F and $\partial_t F$ such that*

$$\|\partial_t u\|_{\mathcal{W}^{k,p}(Z')} \le a_k.$$

(ii) *If $\partial_s F \in \mathcal{W}^{k-1,p}(Z)$ then there is a constant b_k depending on p, μ_0, T, T', $\|\Gamma\|_{C^{2k+2}}$, and the $\mathcal{W}^{k-1,p}(Z)$ norms of F, $\partial_t F$, and $\partial_s F$ such that*

$$\|\partial_s u\|_{\mathcal{W}^{k,p}(Z')} \le b_k.$$

(iii) *If $\partial_t\partial_t F \in \mathcal{W}^{k-1,p}(Z)$ then there is a constant c_k depending on p, μ_0, T, T', $\|\Gamma\|_{C^{2k+2}}$, and the $\mathcal{W}^{k-1,p}(Z)$ norms of F, $\partial_t F$, and $\partial_t\partial_t F$ such that*

$$\|\partial_t\partial_t u\|_{\mathcal{W}^{k,p}(Z')} \le c_k.$$

Notation. In the proofs of lemma 5.1 and proposition 5.3 we use the following notation. The parabolic $\mathcal{C}^k$ norm of a smooth function u is given by

$$\|u\|_{\mathcal{C}^k} := \sum_{2\nu+\mu\le 2k} \|\partial_s^\nu \partial_t^\mu u\|_\infty \,. \tag{15}$$

Compare this to standard space C^k with norm $\|u\|_{C^k} = \sum_{\nu+\mu\le k} \|\partial_s^\nu \partial_t^\mu u\|_\infty$. Given two constants $T > T' > 0$ consider the sequence

$$T_k := T' + \frac{T-T'}{k}, \quad k \in \mathbb{N}. \tag{16}$$

Note that $T_1 = T$. This definition also makes sense if we replace k by a real number $r \ge 1$. Now consider the cylinders $Z_r = (-T_r, 0] \times S^1$. By $\operatorname{int} Z_r$ we denote the interior $(-T_r, 0) \times S^1$ of Z_r. It is useful to memorize that $Z_{r+1} \subset Z_r$. For each positive integer k fix a smooth compactly supported cutoff function

$$\rho_k : (-T_k, 0] \to [0,1] \tag{17}$$

such that $\rho_k = 1$ on Z_{k+1} and $\|\partial_s \rho\|_\infty \ge 1$.

Proof of lemma 5.1. Denote the nonlinear part of the heat equation (14) by

$$h = h(u) = \Gamma(u)\,(\partial_t u, \partial_t u) + F$$

and the first cutoff function fixed in (17) by $\rho = \rho_1$. Then $h \in L^p(Z_2)$, namely

$$\begin{aligned}
\|h\|_{L^p(Z_2)} &\le \left\|\rho^2 h\right\|_{L^p(Z_1)} \\
&\le \|\Gamma\|_\infty \, \||\partial_t(\rho u)| \cdot |\partial_t(\rho u)|\|_{L^p(Z_1)} + \left\|\rho^2 F\right\|_{L^p(Z_1)} \\
&\le C_p \|\Gamma\|_\infty \|\partial_s \rho\|_\infty^2 \|u\|^2_{\mathcal{W}^{1,p}(Z)} + \|F\|_{L^p(Z)}
\end{aligned}$$

where in step one and two we used that $\rho^2 = 1$ on Z_2 and independence of ρ on the t variable, respectively. The last step is by the product estimate theorem 1.1 with constant $C_p > 0$ applied to the compactly supported $\mathcal{W}^{1,p}$ map $\rho u : Z \to \mathbb{R}^N$ using density. Compactness of M implies that $\|\Gamma\|_\infty < \infty$. Next observe that

$$\partial_t h = d\Gamma(u)\,(\partial_t u, \partial_t u, \partial_t u) + 2\Gamma(u)\,(\partial_t \partial_t u, \partial_t u) + \partial_t F. \tag{18}$$

Now we indicate the main idea of proof. Suppose we knew that $\partial_t h \in L^\chi(Z_{k+1})$ for some $\chi > 1$ and some $k \in \mathbb{N}$, then

$$\begin{aligned}
\int_{Z_{k+1}} \partial_t u \,(-\partial_s \phi - \partial_t \partial_t \phi) &= -\int_{Z_{k+1}} \partial_s u \,\partial_t \phi + \int_{Z_{k+1}} \partial_t \partial_t u \,\partial_t \phi \\
&= -\int_{Z_{k+1}} h\,\partial_t \phi \\
&= \int_{Z_{k+1}} \partial_t h\,\phi
\end{aligned} \tag{19}$$

for every $\phi \in C_0^\infty(\operatorname{int} Z_{k+1})$. Here all steps use integration by parts. Step two is by definition of h and the assumption that u satisfies the heat equation (14) almost everywhere. Theorem 1.3 on interior regularity then asserts that $\partial_t u \in \mathcal{W}^{1,\chi}(Z_{k+2})$. Hence we have improved the regularity of $\partial_t u$ which in turn improves the of regularity $\partial_t h$ as given by (18). Now start over again. We prove below that under this iteration χ eventually converges to p. But $p > 2$, hence continuity of $\partial_t u$ follows by the Sobolev embedding $\mathrm{W}^{1,\chi} \hookrightarrow C^0$.

To get the iteration started at $k = 1$ we need to first prove that $\partial_t h \in L^\chi(Z_2)$ for some $\chi > 1$. As a first try recall that $u \in \mathcal{W}^{1,p}(Z_1)$ by assumption, therefore the first term in (18) is in $L^{p/3}$ only whereas the second term is in $L^{p/2}$. Hence $\partial_t h \in L^{p/3}$, but $p/3$ is not necessarily larger than 1. Fortunately, using the product estimate theorem 1.1 we can do better. Recall that $p > 2$ is fixed by assumption. Consider the function

$$\chi = \chi_p(q) = \frac{pq}{p+q}$$

and observe that $1/p + 1/q = 1/\chi$. Apply Hölder's inequality to obtain

$$\begin{aligned}
\|\partial_t h\|_{L^\chi(Z_{k+1})} &\le \left\|{\rho_k}^2 \partial_t h\right\|_{L^\chi(Z_k)} \\
&\le \|d\Gamma\|_\infty \left\| |\partial_t(\rho_k u)| \cdot |\partial_t(\rho_k u)| \right\|_{L^p(Z_k)} \|\partial_t u\|_{L^q(Z_k)} \\
&\quad + 2\|\Gamma\|_\infty \|\partial_t \partial_t u\|_{L^p(Z_k)} \|\partial_t u\|_{L^q(Z_k)} + \|\partial_t F\|_{L^\chi(Z_k)} \\
&\le C_p \|d\Gamma\|_\infty \|\partial_s \rho_k\|_\infty^2 \|u\|_{\mathcal{W}^{1,p}(Z)}^2 \|\partial_t u\|_{L^q(Z_k)} \\
&\quad + 2\|\Gamma\|_\infty \|\partial_t \partial_t u\|_{L^p(Z)} \|\partial_t u\|_{L^q(Z_k)} + \|\partial_t F\|_{L^p(Z_k)} \\
&\le \alpha \|\partial_t u\|_{L^q(Z_k)} + \|\partial_t F\|_{L^p(Z)} .
\end{aligned} \tag{20}$$

Here the third step is by the product estimate theorem 1.1 with constant C_p and the constant α in the last line depends on p, μ_0, $\|\Gamma\|_{C^1}$, and ρ_k. We used again one of the cutoff functions in (17) to produce a compactly supported function as required by the product estimate. Consequently the domain shrinks.

Now we start the iteration with initial value $q_1 = p$. Then $\chi(q_1) = p/2 > 1$. Hence $\partial_t h \in L^{p/2}(Z_2)$ by (20) for $k = 1$. Therefore by (19) theorem 1.3 applies for the functions $\partial_t u$ and $f = \partial_t h$ and proves that $\partial_t u \in \mathcal{W}^{1,p/2}_{loc}(Z_2)$ and

$$\begin{aligned}
\|\partial_t u\|_{\mathcal{W}^{1,p/2}(Z_3)} &\le c_2 \left(\|\partial_t h\|_{L^{p/2}(Z_2)} + \mu_0 \right) \\
&\le c_2 \left(\alpha\mu_0 + \|\partial_t F\|_{L^p(Z)} + \mu_0 \right)
\end{aligned} \tag{21}$$

for some constant $c_2 = c_2(p, T_2 - T_3)$. Step two uses (20) for $k = 1$ and $q = p/2$, the fact that $\|\partial_t u\|_{p/2} \le \|\partial_t u\|_p$, and the assumption $\|\partial_t u\|_p \le \mu_0$.
Now there are three cases: If $p > 4$ then we are done by the Sobolev embedding $W^{1,p/2} \hookrightarrow C^0$ on the domain Z_3; see e.g. [MS, App. B.1] for the relevant embedding theorems. If $p < 4$, then the value of $\chi = \chi_p(q_1) = p/2$ is in the interval $(1, 2)$ and in this case there is the Sobolev embedding

$$\mathcal{W}^{1,\chi}(Z_3) \subset W^{1,\chi}(Z_3) \hookrightarrow L^{2\chi/(2-\chi)}(Z_3) = L^{q_2}(Z_3)$$

with constant $C_2 = C_2(p, T_3) > 0$. Here we abbreviated

$$q_2 := \frac{2\chi}{2-\chi} = \frac{2pq_1}{2p + 2q_1 - pq_1} = \frac{2p}{4-p}.$$

Hence $\partial_t u \in L^{2p/(4-p)}(Z_3)$. Since $2p/(4-p) > p$ is equivalent to $2 < p < 4$, this means that the regularity of $\partial_t u$ has been improved – on the expense of a smaller domain though. The case $p = 4$ means that $u : Z \to \mathbb{R}^N$ is a $\mathcal{W}^{1,4}$ map to start with. But then it is also a $\mathcal{W}^{1,3}$ map and we are in the former case.

Repeating the same argument with new initial value q_2 proves that $\partial_t u \in \mathcal{W}^{1,\chi_p(q_2)}(Z_5)$. Again this space embedds either in $C^0(Z_5)$ and we are done or it embedds in $L^{q_3}(Z_5)$ where $q_3 = 2pq_2/(2p + 2q_2 - pq_2) > q_2$. It is crucial that in (20) the value of p is fixed. Firstly, because the product estimate theorem 1.1 requires $p \geq 2$ and, secondly, because we only know that $\partial_t \partial_t u \in L^p$. Proceeding this way we obtain the sequence q_k determined by

$$q_{k+1} = \frac{2pq_k}{2p + 2q_k - pq_k}, \qquad q_1 = p. \tag{22}$$

Observe again that the condition $p > 2$ implies that $q_{k+1} > q_k$. Hence the sequence is strictly monotone increasing. Next we prove that $q_k \to \infty$ as $k \to \infty$. Assume by contradiction that this is not true. Then by strict monotonicity the sequence is bounded and admits a unique limit, say q. By (22) this limit satisfies $q = 2pq/(2p + 2q - pq)$. But this is equivalent to $p = 2$ contradicting $p > 2$. It follows that $\chi_p(q_k)$ converges to p as $k \to \infty$. But $p > 2$, hence whenever k is sufficiently large there is the Sobolev embedding

$$\mathcal{W}^{1,\chi_p(q_k)}(Z_{2k+1}) \hookrightarrow C^0(Z_{2k+1}) \subset C^0(Z')$$

and this implies the estimate in lemma 5.1. Clearly $\partial_t u$ is continuous on the whole cylinder Z since every point is contained in some subcylinder Z'. $\square$

Proof of proposition 5.3. We prove the following claim by induction on ℓ. Recall from (16) the definition of the reals T_ℓ and the cylinders Z_ℓ.

Claim. Given $0 < T' < T$ and $k \geq 1$ such that F and $\partial_t F$ are in $\mathcal{W}^{k-1,p}$, then the following is true for every $\ell \in \{1, \ldots, k\}$.

(a) *$\partial_t u \in \mathcal{W}^{\ell,p}_{loc}(Z_{3\ell-1})$ and there exists a constant A_ℓ depending on p, μ_0, $\|\Gamma\|_{C^{2\ell+2}}$, $\|F\|_{\mathcal{W}^{\ell-1,p}}$, and $\|\partial_t F\|_{\mathcal{W}^{\ell-1,p}}$ such that*

$$\|\partial_t u\|_{\mathcal{W}^{\ell,p}(Z_{3\ell})} \leq A_\ell.$$

(b) *If $\partial_s F \in \mathcal{W}^{k-1,p}$ then $\partial_s u \in \mathcal{W}^{\ell,p}_{loc}(Z_{3\ell})$ and there exists a constant B_ℓ depending on p, μ_0, $\|\Gamma\|_{C^{2\ell+2}}$, $\|F\|_{\mathcal{W}^{\ell-1,p}}$, $\|\partial_t F\|_{\mathcal{W}^{\ell-1,p}}$, and $\|\partial_s F\|_{\mathcal{W}^{\ell-1,p}}$ such that*

$$\|\partial_s u\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+1})} \leq B_\ell.$$

(c) *If* $\partial_t\partial_t F \in \mathcal{W}^{k-1,p}$ *then* $\partial_t\partial_t u \in \mathcal{W}^{\ell,p}_{loc}(Z_{3\ell+1})$ *and there exists a constant* C_ℓ *depending on* p, μ_0, $\|\Gamma\|_{C^{2\ell+2}}$, $\|F\|_{\mathcal{W}^{\ell-1,p}}$, $\|\partial_t F\|_{\mathcal{W}^{\ell-1,p}}$, *and* $\|\partial_t\partial_t F\|_{\mathcal{W}^{\ell-1,p}}$ *such that*

$$\|\partial_t\partial_t u\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+2})} \le C_\ell.$$

Here and throughout the domain of all norms is $Z = Z_T$, unless specified otherwise. An exception are the various norms of Γ appearing below for which the domain is the compact manifold M. We abbreviate $h = \Gamma(u)\,(\partial_t u, \partial_t u) + F$.

Case $\ell = 1$. By lemma 5.1 with $T' = T_2$ there is a constant C_0 depending on p, μ_0, T, T_2, and $\|\Gamma\|_{C^1}$, such that

$$\|\partial_t u\|_{C^0(Z_2)} \le C_0 \left(1 + \|\partial_t F\|_p\right). \tag{23}$$

(a) Recall that $\partial_t h$ is given by (18). Straightforward calculation shows that

$$\begin{aligned}\|\partial_t h\|_{L^p(Z_2)} &\le \|d\Gamma\|_\infty \|\partial_t u\|^2_{C^0(Z_2)} \|\partial_t u\|_{L^p(Z_2)} + \|\partial_t F\|_{L^p(Z_2)} \\ &\quad + 2\|\Gamma\|_\infty \|\partial_t u\|_{C^0(Z_2)} \|\partial_t\partial_t u\|_{L^p(Z_2)} \\ &\le \alpha\left(1 + \|\partial_t F\|_p^2\right)\end{aligned}$$

for some constant $\alpha = \alpha(p, \mu_0, T, T_2, \|\Gamma\|_{C^1})$. We used (23) and the assumption $\|u\|_{\mathcal{W}^{1,p}} \le \mu_0$. Recall from (19) that $\partial_t u$ satisfies

$$\int_{Z_2} \partial_t u\,(-\partial_s\phi - \partial_t\partial_t\phi) = \int_{Z_2} \partial_t h\,\phi$$

for every $\phi \in C_0^\infty(\operatorname{int} Z_2)$. Hence theorem 1.3 on interior regularity for $q = p$, $T = T_2$, $T' = T_3$, $k = 0$, and the functions $f = \partial_t h$ and $\partial_t u$ in $L^p(Z_2)$ proves that $\partial_t u \in \mathcal{W}^{1,p}_{loc}(Z_2)$ and

$$\|\partial_t u\|_{\mathcal{W}^{1,p}(Z_3)} \le \mu\left(\|\partial_t h\|_{L^p(Z_2)} + \|\partial_t u\|_{L^p(Z_2)}\right)$$

for some constant $\mu = \mu(p, T_2, T_3)$. Now use the estimate for $\partial_t h$ to see that

$$\|\partial_t u\|_{\mathcal{W}^{1,p}(Z_3)} \le A\left(1 + \|\partial_t F\|_p^2\right)$$

for some constant $A = A(p, \mu_0, T, T_2, T_3, \|\Gamma\|_{C^1})$.

(b) Straightforward calculation shows that

$$\begin{aligned}\|\partial_s h\|_{L^p(Z_3)} &\le \|d\Gamma\|_\infty \|\partial_t u\|^2_{C^0(Z_3)} \|\partial_s u\|_{L^p(Z_3)} + \|\partial_s F\|_{L^p(Z_3)} \\ &\quad + 2\|\Gamma\|_\infty \|\partial_t u\|_{C^0(Z_3)} \|\partial_s\partial_t u\|_{L^p(Z_3)} \\ &\le \beta\left(1 + \|\partial_t F\|_p^3\right) + \|\partial_s F\|_p\end{aligned}$$

for some constant $\beta = \beta(p, \mu_0, T, T_2, T_3, \|\Gamma\|_{C^1}) > 1$. Here we estimated the L^p norm of $\partial_s\partial_t u$ by the $\mathcal{W}^{1,p}$ estimate for $\partial_t u$ just proved in (a). We also used the C^0 estimate (23). Next observe that

$$\begin{aligned}
\int_{Z_3} \partial_s u \, (-\partial_s\phi - \partial_t\partial_t\phi) &= -\int_{Z_3} (\partial_s u - \partial_t\partial_t u) \, \partial_s\phi \\
&= -\int_{Z_3} (\Gamma(u)\,(\partial_t u, \partial_t u) + F(u)) \, \partial_s\phi \\
&= \int_{Z_3} \partial_s h \, \phi
\end{aligned} \tag{24}$$

for every $\phi \in C_0^\infty(\operatorname{int} Z_3)$. Here steps one and three are by integration by parts. Step two uses the assumption that u satisfies the heat equation (14) almost everywhere. Now theorem 1.3 proves that $\partial_s u \in \mathcal{W}^{1,p}_{loc}(Z_3)$ and

$$\|\partial_s u\|_{\mathcal{W}^{1,p}(Z_4)} \le \mu \left(\|\partial_s h\|_{L^p(Z_3)} + \|\partial_s u\|_{L^p(Z_3)} \right)$$

for some constant $\mu = \mu(p, T_3, T_4)$. Now use the estimate for $\partial_s h$ to see that

$$\|\partial_s u\|_{\mathcal{W}^{1,p}(Z_4)} \le B \left(1 + \|\partial_t F\|_p^3 + \|\partial_s F\|_p \right)$$

for some constant $B = B(p, \mu_0, T, T_2, T_3, T_4, \|\Gamma\|_{C^1})$.
(c) Straighforward calculation shows that

$$\begin{aligned}
\|\partial_t\partial_t h\|_{L^p(Z_4)} \le& \left\|d^2\Gamma\right\|_\infty \|\partial_t u\|^3_{C^0(Z_4)} \|\partial_t u\|_{L^p(Z_4)} + \|\partial_t\partial_t F\|_{L^p(Z_4)} \\
&+ 4\,\|d\Gamma\|_\infty \|\partial_t u\|^2_{C^0(Z_4)} \|\partial_t\partial_t u\|_{L^p(Z_4)} \\
&+ 2\,\|\Gamma\|_\infty \|\partial_t u\|_{C^0(Z_4)} \|\partial_t\partial_t\partial_t u\|_{L^p(Z_4)} \\
&+ 2\,\|\Gamma\|_\infty \|\partial_t\partial_t u\|_{C^0(Z_4)} \|\partial_t\partial_t u\|_{Lp(Z_4)} \\
\le& \,\gamma \left(1 + \|\partial_t F\|_p^4 \right) + \|\partial_t\partial_t F\|_p
\end{aligned}$$

for some constant $\gamma = \gamma(p, \mu_0, T, T_2, T_3, T_4, \|\Gamma\|_{C^2})$. In the final inequality we used the C^0 estimate (23) for $\partial_t u$ and the $\mathcal{W}^{1,p}$ estimate for $\partial_t u$ proved above in (a). This takes care of all terms but one, namely the C^0 norm of $\partial_t\partial_t u$. Here we use that $\partial_t\partial_t\partial_t u$ and $\partial_s\partial_t\partial_t u = \partial_t\partial_t\partial_s u$ are in $L^p(Z_4)$ by (a) and (b), respectively. Hence $\partial_t\partial_t u \in C^0$ by the Sobolev embedding $W^{1,p} \hookrightarrow C^0$. Similarly to the calculation in (19) it follows that

$$\int_{Z_4} \partial_t\partial_t u \, (-\partial_s\phi - \partial_t\partial_t\phi) = \int_{Z_4} \partial_t\partial_t h \, \phi$$

for every $\phi \in C_0^\infty(\operatorname{int} Z_4)$. Theorem 1.3 then proves that $\partial_t\partial_t u \in \mathcal{W}^{1,p}_{loc}(Z_4)$ and

$$\|\partial_t\partial_t u\|_{\mathcal{W}^{1,p}(Z_5)} \le \mu \left(\|\partial_t\partial_t h\|_{L^p(Z_4)} + \|\partial_t\partial_t u\|_{L^p(Z_4)} \right)$$

for some constant $\mu = \mu(p, T_4, T_5)$. Now use the estimate for $\partial_t\partial_t h$ to see that

$$\|\partial_t\partial_t u\|_{\mathcal{W}^{1,p}(Z_5)} \le C\left(1 + \|\partial_t F\|_p^4 + \|\partial_t\partial_t F\|_p\right)$$

for some constant $C = C(p, \mu_0, T, T_2, T_3, T_4, T_5, \|\Gamma\|_{C^2})$.

Induction step $\ell \Rightarrow \ell+1$. Fix an integer $\ell \in \{1, \dots, k-1\}$ and assume that (a–c) are true for this choice of ℓ. We indicate this by the notation $(\text{a–c})_\ell$. The task at hand is to prove $(\text{a–c})_{\ell+1}$. Recall the parabolic $\mathcal{C}^\ell$ norm (15). An immediate consequence of the induction hypothesis $(\text{a–c})_\ell$ is that

$$\|u\|_{\mathcal{W}^{\ell+1,p}(Z_{3\ell+2})} \le D'_{\ell+1}$$

for some constant $D'_{\ell+1} = D'_{\ell+1}(p, \mu_0, \|\Gamma\|_{C^{2\ell+2}}, \|F\|_{\mathcal{W}^{\ell,p}})$. Hence

$$\|u\|_{\mathcal{C}^\ell(Z_{3\ell+2})} \le D_{\ell+1} \tag{25}$$

for some constant $D_{\ell+1} = D_{\ell+1}(p, \mu_0, \|\Gamma\|_{C^{2\ell+2}}, \|F\|_{\mathcal{W}^{\ell,p}})$. To see this observe that up to a constant the $\mathcal{C}^\ell$ norm can be estimated by the $\mathcal{W}^{\ell+1,p}$ norm. (This boils down to the Sobolev embedding $W^{1,p} \hookrightarrow C^0$ for each individual derivative of u showing up in $\mathcal{C}^\ell$.)

$(\text{a})_{\ell+1}$ Straightforward calculation shows that

$$\begin{aligned}
&\|\partial_t h\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+2})} \\
&\le \|d\Gamma\|_{C^{2\ell}}\, d_\ell\, \|u\|^2_{\mathcal{C}^\ell(Z_{3\ell+2})} \|\partial_t u\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+2})} + \|\partial_t F\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+2})} \\
&\quad + 2\,\|\Gamma\|_{C^{2\ell}}\, d_\ell\, \|u\|_{\mathcal{C}^\ell(Z_{3\ell+2})} \left(\|\partial_t u\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+2})} + \|\partial_t\partial_t u\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+2})}\right) \\
&\le \alpha_{\ell+1} + \|\partial_t F\|_{\mathcal{W}^{\ell,p}}
\end{aligned}$$

for some constant $\alpha_{\ell+1} = \alpha_{\ell+1}(p, \mu_0, \|\Gamma\|_{C^{2\ell+2}}, \|F\|_{\mathcal{W}^{\ell,p}})$. The first inequality follows from the identity (18) and the last two estimates of corollary 5.5 with constant d_ℓ. Notice the difference between the standard C^ℓ and the parabolic $\mathcal{C}^\ell$ norms. To obtain the second inequality we applied (25) and the induction hypotheses $(\text{a})_\ell$ and $(\text{c})_\ell$ to estimate the $\mathcal{W}^{\ell,p}$ norms of $\partial_t u$ and $\partial_t\partial_t u$, respectively. Next observe that theorem 1.3 applies by (19) and shows that $\partial_t u \in \mathcal{W}^{\ell+1,p}_{loc}(Z_{3\ell+2})$ and

$$\|\partial_t u\|_{\mathcal{W}^{\ell+1,p}(Z_{3\ell+3})} \le \mu\left(\|\partial_t h\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+2})} + \|\partial_t u\|_{L^p(Z_{3\ell+2})}\right)$$

for some constant $\mu = \mu(p, Z_{3\ell+2}, Z_{3\ell+3})$. Now the assumption $\|u\|_{\mathcal{W}^{1,p}} \le \mu_0$ and the estimate for $\partial_t h$ conclude the proof of $(\text{a})_{\ell+1}$. For latter reference we remark that $(\text{a})_{\ell+1}$ implies – similarly to (25) – the estimate

$$\|\partial_t u\|_{\mathcal{C}^\ell(Z_{3\ell+3})} \le E_\ell \tag{26}$$

for some constant $E_\ell = E_\ell(p, \mu_0, \|\Gamma\|_{C^{2\ell+2}}, \|F\|_{\mathcal{W}^{\ell,p}}, \|\partial_t F\|_{\mathcal{W}^{\ell,p}})$.

$(\mathrm{b})_{\ell+1}$ Straightforward calculation using the $\mathcal{W}^{\ell+1,p}$ estimate for $\partial_t u$ just proved and the induction hypotheses $(\mathrm{a–c})_\ell$ implies that

$$\begin{aligned}\|\partial_s h\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+3})} &\le \|d\Gamma\|_{C^{2\ell}} \|\partial_t u\|^2_{\mathcal{C}^\ell(Z_{3\ell+3})} \|\partial_s u\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+3})} + \|\partial_s F\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+3})} \\ &\quad + 2\|\Gamma\|_{C^{2\ell}} \|\partial_t u\|_{\mathcal{C}^\ell(Z_{3\ell+3})} \|\partial_s\partial_t u\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+3})} \\ &\le \beta_{\ell+1} + \|\partial_s F\|_{\mathcal{W}^{\ell,p}}\end{aligned}$$

for some constant $\beta_{\ell+1} = \beta_{\ell+1}(p, \mu_0, \|\Gamma\|_{C^{2\ell+2}}, \|F\|_{\mathcal{W}^{\ell,p}}, \|\partial_t F\|_{\mathcal{W}^{\ell,p}})$. To obtain the first inequality we simply pulled out the $\mathcal{C}^\ell$ norms. In the second inequality we used (26), the induction hypothesis $(\mathrm{b})_\ell$ to estimate the $\mathcal{W}^{\ell,p}$ norm of $\partial_s u$, and the induction hypothesis $(\mathrm{a})_{\ell+1}$ just proved to estimate the $\mathcal{W}^{\ell,p}$ norm of $\partial_s\partial_t u$. Next observe that theorem 1.3 applies by the identity (24) with Z_3 replaced by $Z_{3\ell+3}$ and shows that $\partial_s u \in \mathcal{W}^{\ell+1,p}_{loc}(Z_{3\ell+3})$ and

$$\|\partial_s u\|_{\mathcal{W}^{\ell+1,p}(Z_{3\ell+4})} \le \mu \left(\|\partial_s h\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+4})} + \|\partial_t u\|_{L^p(Z_{3\ell+4})} \right)$$

for some constant $\mu = \mu(p, Z_{3\ell+3}, Z_{3\ell+4})$. Now use the estimate for $\partial_s h$.
$(\mathrm{c})_{\ell+1}$ Straighforward calculation shows that

$$\begin{aligned}\|\partial_t\partial_t h\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+3})} &\le \left\|d^2\Gamma\right\|_{C^{2\ell}} \|\partial_t u\|^3_{\mathcal{C}^\ell} \|\partial_t u\|_{\mathcal{W}^{\ell,p}} \\ &\quad + 5\|d\Gamma\|_{C^{2\ell}} \|\partial_t u\|^2_{\mathcal{C}^\ell} \|\partial_t\partial_t u\|_{\mathcal{W}^{\ell,p}} \\ &\quad + 2\|\Gamma\|_{C^{2\ell}} \|\partial_t u\|_{\mathcal{C}^\ell} \|\partial_t\partial_t\partial_t u\|_{\mathcal{W}^{\ell,p}} + \|\partial_t\partial_t F\|_{\mathcal{W}^{\ell,p}} \\ &\quad + 2\|\Gamma\|_{C^{2\ell}} C'_k \|\partial_t u\|_{\mathcal{C}^\ell} \|\partial_t\partial_t u\|_{\mathcal{W}^{\ell,p}}.\end{aligned}$$

Here all norms are taken on the domain $Z_{3\ell+3}$ except those involving Γ which are taken over M. Notice that in the first three terms of the sum we simply pulled out the $\mathcal{C}^\ell$ norms. However, in the last term there appears originally the product $\partial_t\partial_t u$ times $\partial_t\partial_t u$. To deal with this product we applied the first estimate of corollary 5.5 (where in both factors u is replaced by $\partial_t u$).
Now the $\mathcal{C}^\ell$ estimate (26) for $\partial_t u$ and the $\mathcal{W}^{\ell+1,p}$ estimate for $\partial_t u$ established in $(\mathrm{a})_{\ell+1}$ above prove that

$$\|\partial_t\partial_t h\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+3})} \le \gamma_{\ell+1} + \|\partial_t\partial_t F\|_{\mathcal{W}^{\ell,p}}$$

for some constant $\gamma_{\ell+1} = \gamma_{\ell+1}(\ell, p, \mu_0, \|\Gamma\|_{C^{2\ell+2}}, \|F\|_{\mathcal{W}^{\ell,p}}, \|\partial_t F\|_{\mathcal{W}^{\ell,p}})$. Apply again theorem 1.3 to see that $\partial_t\partial_t u \in \mathcal{W}^{\ell+1,p}_{loc}(Z_{3\ell+3})$ and

$$\|\partial_t\partial_t u\|_{\mathcal{W}^{\ell+1,p}(Z_{3\ell+4}))} \le \mu \left(\|\partial_t\partial_t h\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+3})} + \|\partial_t\partial_t u\|_{L^p(Z_{3\ell+3})} \right)$$

for some constant $\mu = \mu(p, Z_{3\ell+3}, Z_{3\ell+4})$. The estimate for $\partial_t\partial_t h$ then proves (c) in the case $\ell + 1$. This completes the proof of the induction step and therefore of the claim. The claim with $\ell = k$ proves proposition 5.3. $\square$

The following product estimates have been used in the parabolic bootstrap iteration above. Recall the definition (15) of the parabolic $\mathcal{C}^k$ norm.

Lemma 5.4. *Fix a constant $p > 2$ and a bounded open subset $\Omega \subset \mathbb{R}^2$ with area $|\Omega|$. Then for every integer $k \geq 1$ there is a constant $c = c(k, |\Omega|)$ such that*

$$\|\partial_t u \cdot v\|_{\mathcal{W}^{k,p}} \leq c \left(\|\partial_t u\|_{\mathcal{W}^{k,p}} \|v\|_\infty + \|u\|_{\mathcal{C}^k} \|v\|_{\mathcal{W}^{k,p}}\right)$$

for all functions $u, v \in C^\infty(\overline{\Omega})$.

Proof. The proof is by induction on k. By definition of the $\mathcal{W}^{\ell,p}$ norm

$$\begin{aligned} \|\partial_t u \cdot v\|_{\mathcal{W}^{\ell+1,p}} \leq & \|\partial_t u \cdot v\|_{\mathcal{W}^{\ell,p}} + \|\partial_t\partial_t u \cdot v + \partial_t u \cdot \partial_t v\|_{\mathcal{W}^{\ell,p}} \\ & + \|\partial_t\partial_t\partial_t u \cdot v + 2\partial_t\partial_t u \cdot \partial_t v + \partial_t u \cdot \partial_t\partial_t v\|_{\mathcal{W}^{\ell,p}} \\ & + \|\partial_s\partial_t u \cdot v + \partial_t u \cdot \partial_s v\|_{\mathcal{W}^{\ell,p}} . \end{aligned} \tag{27}$$

Case $k = 1$. Estimate (27) for $\ell = 0$ shows that

$$\begin{aligned} \|\partial_t u \cdot v\|_{\mathcal{W}^{1,p}} \leq & \left(\|\partial_t u\|_p + \|\partial_t\partial_t u\|_p + \|\partial_t\partial_t\partial_t u\|_p + \|\partial_s\partial_t u\|_p\right) \|v\|_\infty \\ & + \left(\|\partial_t u\|_\infty + 2\|\partial_t\partial_t u\|_\infty\right) \|\partial_t v\|_p \\ & + \|\partial_t u\|_\infty \left(\|\partial_t\partial_t v\|_p + \|\partial_s v\|_p\right) \end{aligned}$$

and this proves the lemma for $k = 1$.

Induction step $k \Rightarrow k + 1$. Consider estimate (27) for $\ell = k$, then inspect the right hand side term by term using the induction hypothesis to conclude the proof. To illustrate this we give full details for the last term in (27), namely

$$\begin{aligned} \|\partial_t u \cdot \partial_s v\|_{\mathcal{W}^{k,p}} & \leq c \left(\|\partial_t u\|_{\mathcal{W}^{k,p}} \|\partial_s v\|_\infty + \|u\|_{\mathcal{C}^k} \|\partial_s v\|_{\mathcal{W}^{k,p}}\right) \\ & \leq c \left(c' |\Omega| \|\partial_t u\|_{\mathcal{C}^k} \|\partial_s v\|_{\mathcal{W}^{1,p}} + \|u\|_{\mathcal{C}^k} \|v\|_{\mathcal{W}^{k+1,p}}\right) \\ & \leq c \left(c' |\Omega| \|u\|_{\mathcal{C}^{k+1}} \|v\|_{\mathcal{W}^{2,p}} + \|u\|_{\mathcal{C}^k} \|v\|_{\mathcal{W}^{k+1,p}}\right) . \end{aligned}$$

Step one is by the induction hypothesis. In step two we pulled out the L^∞ norms of all derivatives of $\partial_t u$ and for the term $\partial_s v$ we used the Sobolev embedding $\mathcal{W}^{1,p} \subset W^{1,p} \hookrightarrow C^0$ with constant c'. Here the assumptions $p > 2$ and Ω bounded enter. Step three is obvious. Now $\mathcal{W}^{k+1,p} \hookrightarrow \mathcal{W}^{2,p}$ since $k \geq 1$. □

Corollary 5.5. *Fix a constant $p > 2$ and a bounded open subset $\Omega \subset \mathbb{R}^2$. Then for every integer $k \geq 1$ there is a constant $d = d(k, |\Omega|)$ such that*

$$\begin{aligned} \|\partial_t u \cdot \partial_t u\|_{\mathcal{W}^{k,p}} & \leq d_k \|u\|_{\mathcal{C}^k} \|\partial_t u\|_{\mathcal{W}^{k,p}} \\ \|\partial_t u \cdot \partial_t\partial_t u\|_{\mathcal{W}^{k,p}} & \leq d_k \|u\|_{\mathcal{C}^k} \left(\|\partial_t u\|_{\mathcal{W}^{k,p}} + \|\partial_t\partial_t u\|_{\mathcal{W}^{k,p}}\right) \\ \|\partial_t u \cdot \partial_t u \cdot \partial_t u\|_{\mathcal{W}^{k,p}} & \leq d_k \|u\|_{\mathcal{C}^k}^2 \|\partial_t u\|_{\mathcal{W}^{k,p}} \end{aligned}$$

for every function $u \in C^\infty(\overline{\Omega})$.

Proof. All three estimates follow from lemma 5.4. To obtain the first and the second estimate set $v = \partial_t u$ and $v = \partial_t\partial_t u$, respectively, and use that

$$\|\partial_t u\|_\infty \leq \|u\|_{\mathcal{C}^k} , \qquad \|\partial_t\partial_t u\|_\infty \leq \|u\|_{\mathcal{C}^k} .$$

To obtain the third estimate set $v = \partial_t u \cdot \partial_t u$ and use in addition the first estimate of corollary 5.5. □

Proof of theorem 1.4. The $\mathcal{W}^{k+1,p}$ norm of u is equivalent to the sum of the $\mathcal{W}^{k,p}$ norms of u, $\partial_t u$, $\partial_s u$, and $\partial_t\partial_t u$. Apply proposition 5.3 (i–iii). □

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